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Galois symbol maps for abelian varieties over a p-adic field

by

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1. Introduction. Let k be a *p-adic field*, that is, a finite extension of $\mathbb{Q}_p$, and denote its residue field by $\mathbb{F}$. An objective of the class field theory of a (projective smooth and geometrically connected) curve X over k with function field $k(X)$ ([1], [15]) is to describe the abelian fundamental group $\pi_1^{\mathrm{ab}}(X)$ by means of the abelian group

$$SK_1(X) = \mathrm{Coker}\Big(K_2(k(X)) \to \bigoplus_{x\in X_0} k(x)^\times\Big)$$

(for the precise definition of $SK_1(X)$, see (2.7)) through the reciprocity map

$$\rho : SK_1(X) \to \pi_1^{\mathrm{ab}}(X)$$

(see Thm. 2.4). The "geometric part" $\mathrm{Ker}(\pi_1^{\mathrm{ab}}(X) \to \pi_1^{\mathrm{ab}}(\mathrm{Spec}(k)))$ denoted by $\pi_1^{\mathrm{ab}}(X)^{\mathrm{geo}}$ is approximated by

$$V(X) = \mathrm{Ker}(\partial : SK_1(X) \to k^\times),$$

where ∂ is defined by the tame symbols. It is known that the induced map $\tau : V(X) \to \pi_1^{\mathrm{ab}}(X)^{\mathrm{geo}}$ from ρ has finite image and its kernel is the maximal divisible subgroup $V(X)_{\mathrm{div}}$ of $V(X)$. In particular, we have a decomposition

$$V(X) = V(X)_{\mathrm{div}} \oplus V(X)_{\mathrm{fin}},$$

for some finite group $V(X)_{\mathrm{fin}}$. Now, we further assume that X has *good reduction and also* $X(k) \neq \emptyset$. Recall that X said to have *good reduction* if the special fiber $\overline{\mathscr{X}} := \mathscr{X} \otimes_{O_k} \mathbb{F}$ of the regular model $\mathscr{X}$ over O_k with $\mathscr{X} \otimes_{O_k} k \simeq X$ is also a smooth curve over $\mathbb{F}$. The Jacobian variety $\mathrm{Jac}(\mathscr{X})$ associated to $\mathscr{X}$ has generic fiber $J = \mathrm{Jac}(X)$ and special fiber $\overline{J} = \mathrm{Jac}(\overline{\mathscr{X}})$. In particular, J has good reduction. The group $V(X)_{\mathrm{fin}}$ is related to the

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G_k-coinvariant part of the Tate module $T(J) = \varprojlim_m J[m]$ via

$$V(X)_{\mathrm{fin}} \xrightarrow{\simeq} T(J)_{G_k}.$$

In this setting, the prime-to-p-part of $V(X)_{\mathrm{fin}}$ is well known as $V(X)/m \simeq \overline{J}(\mathbb{F})/m$ for any m prime to p (by S. Bloch, see Prop. 2.5). For the p-part, only the finiteness is proved [1, Prop. 2.4].

The aim of this note is *to study* $V(X)_{\mathrm{fin}}$ *more explicitly by determining the group structure* under the following conditions:

(**Rat**) $J[p] \subset J(k)$,
(**Ord**) $\overline{J}$ has ordinary reduction, and
(**Ram**) $k(\mu_{p^{N+1}})/k$ is a non-trivial totally ramified extension, where $\mu_{p^{N+1}}$ is the group of p^{N+1}th roots of unity, and

$$N = \max\{n \mid J[p^n] \subset J(k)\}.$$

The main result of this note is the following:

Theorem 1.1 (Cor. 4.9). *Under the conditions* (**Rat**), (**Ord**), *and* (**Ram**) *for* $J = \mathrm{Jac}(X)$ *as above, we have*

$$V(X)_{\mathrm{fin}} \simeq (\mathbb{Z}/p^N)^{\oplus g} \oplus \overline{J}(\mathbb{F}), \quad \textit{where} \quad g = \dim J.$$

More precisely, we show the above theorem under the condition that the Jacobian variety J has *good ordinary reduction*, in the sense that J has good reduction and its reduction $\overline{J}$ has ordinary reduction, without assuming X has good reduction. Note that if X has good reduction, then the Jacobian variety $J = \mathrm{Jac}(X)$ also has good reduction. But the converse does not hold. Since the divisible part $V(X)_{\mathrm{div}}$ is the kernel of the reciprocity map $\tau : V(X) \to \pi_1^{\mathrm{ab}}(X)^{\mathrm{geo}}$ induced from ρ and the map τ is surjective when J has good reduction, we obtain the structure of the geometric part of the fundamental group as follows:

$$\pi_1^{\mathrm{ab}}(X)^{\mathrm{geo}} \simeq (\mathbb{Z}/p^N)^{\oplus g} \oplus \overline{J}(\mathbb{F}).$$

The key tool to compute the group $V(X)$ is the so called *Galois symbol map*

$$s_m : K(k; \mathbb{G}_{\mathrm{m}}, J)/m \to H^2(k, \mu_m \otimes J[m]),$$

where $K(k; \mathbb{G}_{\mathrm{m}}, J)$ is the Somekawa K-group which is isomorphic to $V(X)$ and the map s_m is constructed in a similar way to the Galois symbol map on the Milnor K-group $K_2(k)/m \to H^2(k, \mu_m^{\otimes 2})$ (for the precise construction of the map s_m, see Def. 2.3).

Finally, we give some remarks on the conditions above. The condition (**Rat**) implies $\mu_p \subset k$. In particular, we have $e_k \geq p-1$, where e_k is the absolute ramification index of k. The above theorem should be compared to the following theorem which treated the case where the base field has low ramification:

THEOREM 1.2 (Kato–Saito, Yoshida, see Thm. 4.10). *Assume that the absolute ramification index of k is $e_k < p-1$. Then*

$$V(X)_{\mathrm{fin}} \simeq \overline{J}(\mathbb{F}).$$

The condition (**Ram**) is technical. For example, let us consider an elliptic curve $X = E$ over $k = \mathbb{Q}_p(\mu_{p^M})$ for some $M \geq 1$ and assume that E has good ordinary reduction. Note that $\mathrm{Jac}(E) = E$. From the Weil pairing, we have $N \leq M$, where $N = \max\{n \mid E[p^n] \subset E(k)\}$. Thus, if $E[p^M] \subset E(k)$ then $N = M$ and the condition (**Ram**) above automatically holds. For $M = 1$ (from the above argument, (**Rat**) implies (**Ram**)) and $p = 3$, by using SAGE [14], there are 683 elliptic curves E_0 over $\mathbb{Q}$ with good ordinary reduction at $p = 3$, $\overline{E_0}(\mathbb{F}_3)[3] \neq 0$ and conductor < 1000. Among them, 269 curves satisfy $E[3] \subset E(k)$ and hence the condition (**Ram**), where $E = E_0 \otimes_{\mathbb{Q}} k$ is the base change of E to $k = \mathbb{Q}_3(\mu_3)$.

Content. The content of this note is the following:

- Section 2: After recalling the definition of the Mackey functor and of the product of Mackey functors, we review the class field theory for curves over p-adic fields following Bloch [1] and Saito [15].
- Section 3: We determine the image of the Kummer map $A(k)/p^n \to H^1(k, A[p^n])$ (see (2.4)) associated to an abelian variety A with good ordinary reduction over k (Prop. 3.1). This extends the main theorem in [21] for an elliptic curve. The proof is essentially same as in *op. cit.*
- Section 4: We show that the Galois symbol map $s_{p^n} : K(k; \mathbb{G}_{\mathrm{m}}, A)/p^m \to H^2(k, \mu_{p^n} \otimes A[p^n])$ (defined in Def. 2.3) associated to the multiplicative group $\mathbb{G}_{\mathrm{m}}$ and an abelian variety A over a p-adic field k is bijective (Thm. 4.2), and determine the group structure of the Somekawa K-group $K(k; \mathbb{G}_{\mathrm{m}}, A)$ (Thm. 4.8). Using the isomorphism $K(k; \mathbb{G}_{\mathrm{m}}, J) \simeq V(X)$ (see (2.9)) for a curve X as in Theorem 1.1 and $J = \mathrm{Jac}(X)$, we obtain the structure of $V(X)_{\mathrm{fin}}$.

Notation. Throughout this note, we use the following notation:

- k: a finite extension of $\mathbb{Q}_p$,
- $\mathbb{F}$: the residue field of k, and
- $G_k = \mathrm{Gal}(\overline{k}/k)$: the absolute Galois group of k.

For a finite extension K/k, we define

- O_K: the valuation ring of K with maximal ideal $\mathfrak{m}_K$,
- $\mathbb{F}_K = O_K/\mathfrak{m}_K$: the residue field of K,
- $U_K = O_K^{\times}$: the unit group, and
- $U_K^i = 1 + \mathfrak{m}_K^i$: the higher unit group.

For an abelian group G and $m \in \mathbb{Z}_{\geq 1}$, we write $G[m]$ and G/m for the kernel and cokernel of the multiplication by m on G respectively.

2. Class field theory

Mackey functors

Definition 2.1 (see [13, Sect. 3]). A *Mackey functor* $\mathscr{M}$ (over k) (or a G_k*-modulation* in the sense of [12, Def. 1.5.10]) is a contravariant functor from the category of étale schemes over k to the category of abelian groups equipped with a covariant structure for finite morphisms such that

$$\mathscr{M}(X_1 \sqcup X_2) = \mathscr{M}(X_1) \oplus \mathscr{M}(X_2)$$

and if

$$\begin{CD} X' @>{g'}>> X \\ @V{f'}VV @VV{f}V \\ Y' @>{g}>> Y \end{CD}$$

is a Cartesian diagram, then the induced diagram

$$\begin{CD} \mathscr{M}(X') @>{g'_*}>> \mathscr{M}(X) \\ @A{f'^*}AA @AA{f^*}A \\ \mathscr{M}(Y') @>{g_*}>> \mathscr{M}(Y) \end{CD}$$

commutes.

For a Mackey functor $\mathscr{M}$, we denote by $\mathscr{M}(K)$ its value $\mathscr{M}(\mathrm{Spec}(K))$ for a field extension K of k. For any finite extensions $k \subset K \subset L$, the homomorphisms induced by the canonical map $j : \mathrm{Spec}(L) \to \mathrm{Spec}(K)$ are denoted by

$$N_{L/K} := j_* : \mathscr{M}(L) \to \mathscr{M}(K) \quad \text{and} \quad \mathrm{Res}_{L/K} := j^* : \mathscr{M}(K) \to \mathscr{M}(L).$$

The category of Mackey functors forms an abelian category with the following tensor product:

Definition 2.2 (see [6]). For Mackey functors $\mathscr{M}$ and $\mathscr{N}$, their *Mackey product* $\mathscr{M} \otimes \mathscr{N}$ is defined as follows: For any finite field extension k'/k,

$$(\mathscr{M} \otimes \mathscr{N})(k') := \Big(\bigoplus_{K/k' : \text{finite}} \mathscr{M}(K) \otimes_{\mathbb{Z}} \mathscr{N}(K) \Big) / (\mathbf{PF}), \tag{2.1}$$

where $(\mathbf{PF})$ stands for the subgroup generated by elements of the following form for finite field extensions $k' \subset K \subset L$:

$N_{L/K}(x) \otimes y - x \otimes \mathrm{Res}_{L/K}(y)$ with $x \in \mathscr{M}(L)$ and $y \in \mathscr{N}(K)$, and

$x \otimes N_{L/K}(y) - \mathrm{Res}_{L/K}(x) \otimes y$ with $x \in \mathscr{M}(K)$ and $y \in \mathscr{N}(L)$.

For the Mackey product $\mathscr{M} \otimes \mathscr{N}$, we write $\{x, y\}_{K/k'}$ for the image of $x \otimes y \in \mathscr{M}(K) \otimes_{\mathbb{Z}} \mathscr{N}(K)$ in the product $(\mathscr{M} \otimes \mathscr{N})(k')$. For any finite field extension k'/k, the push-forward

$$N_{k'/k} = j_* : (\mathscr{M} \otimes \mathscr{N})(k') \to (\mathscr{M} \otimes \mathscr{N})(k) \tag{2.2}$$

is given by $N_{k'/k}(\{x, y\}_{K/k'}) = \{x, y\}_{K/k}$. For each $m \in \mathbb{Z}_{\geq 1}$, we define a Mackey functor $\mathscr{M}/m$ by

$$(\mathscr{M}/m)(K) := \mathscr{M}(K)/m \tag{2.3}$$

for any finite extension K/k. We have

$$(\mathscr{M}/m \otimes \mathscr{N}/m)(k) \simeq (\mathscr{M} \otimes \mathscr{N})(k)/m = ((\mathscr{M} \otimes \mathscr{N})/m)(k) \quad \text{(see (2.3))}.$$

Every G_k-module M defines a Mackey functor given by the fixed submodule $M(K) := M^{\mathrm{Gal}(\overline{k}/K))}$ denoted by M. Conversely, assume a Mackey functor $\mathscr{M}$ satisfies *Galois descent*, meaning that, for every finite Galois extension L/K, the restriction

$$\mathrm{Res}_{L/K} : \mathscr{M}(K) \xrightarrow{\simeq} \mathscr{M}(L)^{\mathrm{Gal}(L/K)}$$

is an isomorphism. This Mackey functor $\mathscr{M}$ gives a G_k-module $\varinjlim_{K/k} \mathscr{M}(K)$ (see [12, Chap. 1, Sect. 5, Ex. 1]) denoted again by $\mathscr{M}$. For any $m \in \mathbb{Z}_{\geq 1}$, the connecting homomorphism associated to the short exact sequence $0 \to \mathscr{M}[m] \to \mathscr{M} \xrightarrow{m} \mathscr{M} \to 0$ as G_k-modules gives

$$\delta_{\mathscr{M}} : \mathscr{M}(K)/m \hookrightarrow H^1(K, \mathscr{M}[m]), \tag{2.4}$$

which is often called the *Kummer map*.

Definition 2.3 (see [19, Prop. 1.5]). For Mackey functors $\mathscr{M}$ and $\mathscr{N}$ with Galois descent, the *Galois symbol map*

$$s_m^M : (\mathscr{M} \otimes \mathscr{N})(k)/m \to H^2(k, \mathscr{M}[m] \otimes \mathscr{N}[m]) \tag{2.5}$$

is defined by the cup product and the corestriction as follows:

$$s_m^M(\{x, y\}_{K/k}) = \mathrm{Cor}_{K/k}\left(\delta_{\mathscr{M}}(x) \cup \delta_{\mathscr{N}}(y)\right).$$

Somekawa K-group. For two semi-abelian varieties G_1 and G_2 over k, the G_k-modules $G_1(\overline{k})$ and $G_2(\overline{k})$ define Mackey functors with Galois descent which we denote also by G_1 and G_2. The *Somekawa K-group* $K(k; G_1, G_2)$ is a quotient of $(G_1 \otimes G_2)(k)$ (for the definition, see [19], [13]). The Galois symbol map $s_m^M : (G_1 \otimes G_2)(k)/m \to H^2(k, G_1[m] \otimes G_2[m])$ (Def. 2.3) factors through $K(k; G_1, G_2)$ and the induced map

$$s_m : K(k; G_1, G_2)/m \to H^2(k, G_1[m] \otimes G_2[m]) \tag{2.6}$$

is also called the *Galois symbol map*. Somekawa presented a "conjecture" in which the map s_m is injective (for arbitrary field). For the case $G_1 = G_2 = \mathbb{G}_{\mathrm{m}}$, as $K(k; \mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \simeq K_2^M(k)$ the conjecture holds by the Merkurjev–Suslin theorem [11]. Although it holds in some special cases ([23], [24], and [13]), Spieß and Yamazaki disproved it for some tori [20, Prop. 7].

Class field theory. Following [1], [15], we recall the class field theory for a curve over k. Let X be a projective smooth and geometrically connected curve over k. Define

- X_0: the set of closed points in X,
- $k(X)$: the function field of X,
- $k(x)$: the residue field at $x \in X_0$, and
- $k(X)_x$: the completion of $k(X)$ at $x \in X_0$.

We define

$$SK_1(X) := \mathrm{Coker}\Big(\partial : K_2(k(X)) \to \bigoplus_{x \in X_0} k(x)^\times\Big), \tag{2.7}$$

where the map ∂ is given by the direct sum of the boundary maps $K_2(k(X)_x) \to K_1(k(x)) = k(x)^\times$ for $x \in X_0$. Note that the residue field $k(x)$ is a finite extension field of k so that $k(x)$ is also a p-adic field. The reciprocity maps $k(x)^\times \to \pi_1^{\mathrm{ab}}(x)$ of the local class field theory of $k(x)$ for $x \in X_0$ induce the *reciprocity map* $\rho : SK_1(X) \to \pi_1^{\mathrm{ab}}(X)$. The map ρ is compatible with the reciprocity map $\rho_k : k^\times \to G_k^{\mathrm{ab}} = \mathrm{Gal}(k^{\mathrm{ab}}/k) = \pi_1^{\mathrm{ab}}(\mathrm{Spec}(k))$ of the base field k as in the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & V(X) \arrow[r] \arrow[d, "\tau"] & SK_1(X) \arrow[r, "N"] \arrow[d, "\rho"] & k^\times \arrow[r] \arrow[d, "\rho_k"] & 0 \\ 0 \arrow[r] & \pi_1^{\mathrm{ab}}(X)^{\mathrm{geo}} \arrow[r] & \pi_1^{\mathrm{ab}}(X) \arrow[r, "\varphi"] & G_k^{\mathrm{ab}} \arrow[r] & 0 \end{tikzcd} \tag{2.8}$$

where the map φ is induced from the structure map $X \to \mathrm{Spec}(k)$, N is induced from the norm maps $k(x)^\times \to k^\times$ for each $x \in X_0$, and the groups $V(X)$ and $\pi_1^{\mathrm{ab}}(X)^{\mathrm{geo}}$ are defined by exactness. The main theorem of class field theory for X is the following:

Theorem 2.4 ([1], [15]).

(i) $\pi_1^{\mathrm{ab}}(X)/\overline{\mathrm{Im}(\rho)} \simeq \widehat{\mathbb{Z}}^{\oplus r}$ *for some* $r \geq 0$, *where* $\overline{\mathrm{Im}(\rho)}$ *is the topological closure of the image* $\mathrm{Im}(\rho)$ *in* $\pi_1^{\mathrm{ab}}(X)$.

(ii) $\mathrm{Ker}(\rho) = SK_1(X)_{\mathrm{div}}$, *where* $SK_1(X)_{\mathrm{div}}$ *is the maximal divisible subgroup of* $SK_1(X)$.

(iii) $\mathrm{Ker}(\tau) = V(X)_{\mathrm{div}}$, *where* $V(X)_{\mathrm{div}}$ *is the maximal divisible subgroup of* $V(X)$.

(iv) $\mathrm{Im}(\tau)$ *is finite.*

Note that the invariant r above is determined by the special fiber of the Néron model of the Jacobian variety $J = \mathrm{Jac}(X)$. In particular, $r = 0$

if J has good reduction. From the above theorem, the group $V(X)$ has a decomposition

$$V(X) = V(X)_{\mathrm{div}} \oplus V(X)_{\mathrm{fin}},$$

where the reduced part $V(X)_{\mathrm{fin}}$ is a finite subgroup.

From now on, we *assume* $X(k) \neq \emptyset$. The geometric fundamental group $\pi_1^{\mathrm{ab}}(X)^{\mathrm{geo}}$ is expressed in terms of the Tate module $T(J) = \varprojlim_m J[m]$ of the Jacobian variety $J = \mathrm{Jac}(X)$ as follows: $\pi_1^{\mathrm{ab}}(X)^{\mathrm{geo}} \simeq T(J)_{G_k}$, where $T(J)_{G_k}$ is the G_k-coinvariant quotient of $T(J)$ [16, Chap. II, Lem. 3.2]. On the other hand, the group $V(X)$ can be identified with the Somekawa K-group:

$$V(X) \simeq K(k; \mathbb{G}_{\mathrm{m}}, J) \tag{2.9}$$

([19, Thm. 2.1], [13, Rem. 2.4.2(c)]). The reciprocity map τ in the diagram (2.8) coincides with the Galois symbol map associated with $\mathbb{G}_{\mathrm{m}}$ and J as in the following diagram:

$$\begin{tikzcd} V(X)/m \arrow[r, hook, "\tau"] \arrow[d, "\simeq"] & \pi_1^{\mathrm{ab}}(X)^{\mathrm{geo}}/m \arrow[d, "\simeq"] \\ K(k; \mathbb{G}_{\mathrm{m}}, J)/m \arrow[r, "s_m"] & H^2(k, \mu_m \otimes J[m]) \end{tikzcd} \tag{2.10}$$

Here, the right vertical map is induced from $H^2(k, \mu_m \otimes J[m]) \simeq J[m]_{G_k}$, which is given by the local Tate duality theorem. The above diagram is commutative up to sign [1, Thm. 1.14]. From it, we have $V(X)/m \simeq \mathrm{Im}(s_m)$. For the prime-to-$p$-part of $V(X)$, we have the following proposition:

PROPOSITION 2.5 ([1, Prop. 2.29]). *Assume that X has good reduction and $X(k) \neq \emptyset$. Then, for any $m \in \mathbb{Z}_{\geq 1}$ prime to p, we have $V(X)/m \simeq \overline{J}(\mathbb{F})/m$, where $\overline{J}$ is the reduction of J.*

Prime-to-p-part. We extend Proposition 2.5 using the Somekawa K-group to an abelian variety, essentially following Bloch's proof of the above proposition.

PROPOSITION 2.6. *Let A be an abelian variety over k which has good reduction. For any $m \in \mathbb{Z}_{\geq 1}$ prime to p, we have $K(k; \mathbb{G}_{\mathrm{m}}, A)/m \simeq \overline{A}(\mathbb{F})/m$, where $\overline{A}$ is the reduction of A.*

Proof. The kernel of the reduction map $\pi : A(k) \to \overline{A}(\mathbb{F})$ is isomorphic to the formal group $\widehat{A}(\mathfrak{m}_k) =: \widehat{A}(k)$ which has no prime-to-p torsion [4, Prop. C.2.5, Thm. C.2.6]. The reduction map induces $A(k)/m \simeq \overline{A}(\mathbb{F})/m$ for each m prime to p.

First, we define $\psi : A(k)/m \to K(k; \mathbb{G}_{\mathrm{m}}, A)/m$ by $\psi(x) = \{\pi, x\}_{k/k}$, where π is a uniformizer of k. The map does not depend on the choice of π. In fact, for any $u \in O_k^{\times}$, by taking $\xi \in [m]^{-1}(x)$, $K = k(\xi)/k$ is unramified.

By local class field theory, there exists $\mu \in K^\times$ such that $N_{K/k}\mu = u$. By using (**PF**), we have

$$\{u, x\}_{k/k} = \{N_{K/k}\mu, x\}_{k/k} = \{\mu, \operatorname{Res}_{K/k} x\}_{K/k} = \{\mu, m\xi\}_{k/k} = 0$$

in $K(k; \mathbb{G}_{\mathrm{m}}, A)/m$.

We show that the map ψ is surjective. Take an element of the form $\{\mu\varpi^n, \xi\}_{K/k}$ in $K(k; \mathbb{G}_{\mathrm{m}}, A)/m$, where ϖ is a uniformizer in K, and $\mu \in O_K^\times$. As above, we have $\{\mu, \xi\}_{K/k} = 0$ by considering the unramified extension $K([m]^{-1}\xi)/K$. It is enough to show that $\{\varpi, \xi\}_{K/k}$ is generated by elements of the form $\{\pi, x\}_{k/k}$. Let k' be the maximal unramified subextension of K/k. Then $\pi' = \operatorname{Res}_{k'/k} \pi$ is also a uniformizer of k'. The extension K/k' is totally ramified so that we may take ϖ as $N_{K/k'}\varpi = \pi'$. Since the restriction $\operatorname{Res}_{K/k'} : A(k')/m \xrightarrow{\simeq} A(K)/m$ is bijective, there exists $x' \in A(k')/m$ such that $\operatorname{Res}_{K/k'} x' = \xi$. Therefore,

$$\begin{aligned}
\{\varpi, \xi\}_{K/k} &= \{\varpi, \operatorname{Res}_{K/k'} x'\}_{K/k} \\
&= \{N_{K/k'}\varpi, x'\}_{k'/k} \qquad (\text{by } (\mathbf{PF})) \\
&= \{\operatorname{Res}_{k'/k} \pi, x'\}_{k'/k} \\
&= \{\pi, N_{k'/k}x'\}_{k/k} \qquad (\text{by } (\mathbf{PF})).
\end{aligned}$$

From these equalities, the map ψ is surjective.

Next, we show the following claim: For any m prime to p, we have $\#K(k; \mathbb{G}_{\mathrm{m}}, A)/m \geq \#\overline{A}(\mathbb{F})/m$. The direct limit of the Galois symbol map

$$\varprojlim s_m : K(k; \mathbb{G}_{\mathrm{m}}, A) \to \varprojlim_{m \geq 1} H^2(k, \mu_m \otimes A[m]) \simeq T(A)_{G_k}$$

is known to be surjective [2, Thm. A.1], where $T(A)_{G_k}$ is the G_k-coinvariant quotient of the Tate module $T(A) = \varprojlim_m A[m]$ of A and the latter isomorphism follows from the local Tate duality theorem. Write $T(A) = T_p(A) \times T'(A)$, where $T'(A) = \varprojlim_{(m,p)=1} A[m]$. As A has good reduction, the inertia subgroup $I \subset G_k$ acts trivially on $T'(A)$ so that $T'(A)_{G_k} \simeq T'(\overline{A})_{G_{\mathbb{F}}}$, where $T'(\overline{A}) = \varprojlim_{(m,p)=1} \overline{A}[m]$. The Weil conjecture for abelian varieties implies that $T'(\overline{A})_{G_{\mathbb{F}}}$ is isomorphic to the prime-to-p-part of the torsion subgroup of $\overline{A}(\mathbb{F})$ ([1, Prop. 2.4], [8, Thm. 1(ter)]). For any m prime to p, we have

$$\#K(k; \mathbb{G}_{\mathrm{m}}, A)/m \geq \#T(A)_{G_k}/m = \#T'(\overline{A})_{G_{\mathbb{F}}}/m = \#\overline{A}(\mathbb{F})/m.$$

From the above claim, the surjective homomorphism

$$\overline{A}(\mathbb{F})/m \simeq A(k)/m \overset{\psi}{\twoheadrightarrow} K(k; \mathbb{G}_{\mathrm{m}}, A)/m$$

is bijective by comparing the cardinalities. ∎

3. Kummer map. In this section, let A be an abelian variety of dimension g over k *assuming*

(**Ord**) A has good ordinary reduction, and
(**Rat**) $A[p] \subset A(k)$.

Note that the condition (**Rat**) implies $\mu_p \subset k$ by the Weil pairing [4, Exer. A.7.8]. Let k^{ur} be the completion of the maximal unramified extension of k. The kernel of the reduction is

$$A_1(k^{\mathrm{ur}}) = \mathrm{Ker}(\pi : A(k^{\mathrm{ur}}) \to \overline{A}(\overline{\mathbb{F}})) \simeq \widehat{A}(\mathfrak{m}_{k^{\mathrm{ur}}}) =: \widehat{A}(k^{\mathrm{ur}}),$$

where $\widehat{A}(\mathfrak{m}_{k^{\mathrm{ur}}})$ is the group associated to the formal group $\widehat{A}$ of A (see [4, Thm. C.2.6]). It is known that $\widehat{A} \times_{O_k} \mathrm{Spf}(O_{k^{\mathrm{ur}}}) \simeq (\widehat{\mathbb{G}}_m)^{\oplus g}$, where $\widehat{\mathbb{G}}_m$ is the multiplicative group [10, Lem. 4.26, Lem. 4.27]. Since $A[p] \subset A(k)$, we have $A_1[p] \subset A_1(k)$ and hence we obtain isomorphisms

$$A_1[p] = A_1(k^{\mathrm{ur}})[p] \simeq \widehat{A}(k^{\mathrm{ur}})[p] \simeq ((\widehat{\mathbb{G}}_m)(k^{\mathrm{ur}})[p])^{\oplus g} \simeq (\mu_p)^{\oplus g}. \tag{3.1}$$

Now, we choose an isomorphism

$$A[p] \xrightarrow{\simeq} (\mu_p)^{\oplus 2g} \tag{3.2}$$

of (trivial) Galois modules which makes the following diagram commutative:

$$\begin{tikzcd}
A_1[p] \arrow[r, hook] \arrow[d, "\simeq"] & A[p] \arrow[d, "\simeq"] \\
(\mu_p)^{\oplus g} \arrow[r, hook, "(\mathrm{id},1)"] & (\mu_p)^{\oplus g} \oplus (\mu_p)^{\oplus g}
\end{tikzcd}$$

where the left vertical map is given in (3.1), and the bottom horizontal map is $(\mu_p)^{\oplus g} \to (\mu_p)^{\oplus 2g}$, $(x_1, \dots, x_g) \mapsto (x_1, \dots, x_g, 1, \dots, 1)$. The Kummer map on $\mathbb{G}_{\mathrm{m}}$ gives the isomorphism $\delta_{\mathbb{G}_{\mathrm{m}}} : K^{\times}/p \xrightarrow{\simeq} H^1(K, \mu_p)$ for an extension K/k. In the following, we identify these groups. The fixed isomorphism (3.2) induces an isomorphism (♣) in

$$\delta_A^K : A(K)/p \overset{\delta_A}{\hookrightarrow} H^1(K, A[p]) \overset{(♣)}{\simeq} H^1(K, \mu_p)^{\oplus 2g} = (K^{\times}/p)^{\oplus 2g}.$$

Proposition 3.1. *For any finite extension K/k, we have the following:*

(i) *The image of the Kummer map δ_A^K equals to*

$$(\overline{U}_K)^{\oplus g} \oplus \mathrm{Ker}\big(K^{\times}/p \xrightarrow{j} (K^{\mathrm{ur}})^{\times}/p\big)^{\oplus g},$$

where $\overline{U}_K := \mathrm{Im}(U_K \to K^{\times}/p)$, K^{ur} is the completion of the maximal unramified extension of K, and j is the map induced from the inclusion $K^{\times} \hookrightarrow (K^{\mathrm{ur}})^{\times}$.

(ii) *The image of the composition $A_1(K)/p \to A(K)/p \xrightarrow{\delta_A^K} (K^{\times}/p)^{\oplus 2g}$ coincides with $(\overline{U}_K)^{\oplus g}$. In particular,*

$$\widehat{A}(K)/p \simeq A_1(K)/p \simeq (\overline{U}_K)^{\oplus g}.$$

In the following, we fix a finite extension K/k and prove the above proposition. First, we show the following lemma on

$$\delta_{A_1}^{K^{\mathrm{ur}}} : A_1(K^{\mathrm{ur}})/p \hookrightarrow H^1(K^{\mathrm{ur}}, A_1[p]) \overset{(\diamondsuit)}{\simeq} ((K^{\mathrm{ur}})^\times/p)^{\oplus g},$$

where the isomorphism $(\diamondsuit)$ is given by (3.1).

Lemma 3.2.

(i) $\mathrm{Im}(\delta_{A_1}^{K^{\mathrm{ur}}}) \subset (\overline{U}_{K^{\mathrm{ur}}})^{\oplus g}$.
(ii) $\mathrm{Im}(\delta_{A}^{K^{\mathrm{ur}}}) \subset (\overline{U}_{K^{\mathrm{ur}}})^{\oplus g}$.

Proof. (i) Recall that $A_1(K^{\mathrm{ur}}) \simeq \widehat{A}(K^{\mathrm{ur}}) \simeq (\widehat{\mathbb{G}}_m(K^{\mathrm{ur}}))^{\oplus g}$ as noted above. The isomorphism (3.1) gives the following commutative diagram:

$$\begin{tikzcd}
(\widehat{\mathbb{G}}_m(K^{\mathrm{ur}}))^{\oplus g} & (\mu_p)^{\oplus g} \arrow[l, hook'] \\
A_1(K^{\mathrm{ur}}) \arrow[u, "\simeq"] & A_1[p] \arrow[l, hook'] \arrow[u, "\simeq"']
\end{tikzcd} \tag{3.3}$$

The above diagram makes the square $(\spadesuit)$ commutative in the diagram

$$\begin{tikzcd}
A_1(K^{\mathrm{ur}})/p \arrow[r, "\simeq"] \arrow[d, hook, "\delta_{A_1}"'] \arrow[rr, bend left, "\delta_{A_1}^{K^{\mathrm{ur}}}"] \arrow[dr, phantom, "(\spadesuit)"] & \left(\widehat{\mathbb{G}}_m(K^{\mathrm{ur}})/p\right)^{\oplus g} \arrow[r, "\iota"] \arrow[d, hook, "\delta_{\widehat{\mathbb{G}}_m}"'] & ((K^{\mathrm{ur}})^\times/p)^{\oplus g} \arrow[r, "v"] \arrow[d, equal, "\delta_{\mathbb{G}_m}"] & (\mathbb{Z}/p)^{\oplus g} \\
H^1(K^{\mathrm{ur}}, A_1[p]) \arrow[r, "\simeq"] & H^1(K^{\mathrm{ur}}, \widehat{\mathbb{G}}_m[p])^{\oplus g} \arrow[r, phantom, "="] & H^1(K^{\mathrm{ur}}, \mu_p)^{\oplus g} &
\end{tikzcd}$$

where ι is induced from $\widehat{\mathbb{G}}_m(K^{\mathrm{ur}}) = U^1_{K^{\mathrm{ur}}} \hookrightarrow (K^{\mathrm{ur}})^\times$ and v is the valuation map. Since $\widehat{\mathbb{G}}_m(K^{\mathrm{ur}}) = U^1_{K^{\mathrm{ur}}} \subset O^\times_{K^{\mathrm{ur}}}$, $v \circ \iota = 0$ in the above diagram. Hence, $\mathrm{Im}(\delta_{A_1}^{K^{\mathrm{ur}}}) \subset (\overline{U}_{K^{\mathrm{ur}}})^{\oplus g} = \mathrm{Ker}(v : (K^{\mathrm{ur}}/p)^{\oplus g} \to (\mathbb{Z}/p)^{\oplus g})$.

(ii) Consider the short exact sequence

$$A_1(K^{\mathrm{ur}})/p \to A(K^{\mathrm{ur}})/p \to \overline{A}(\mathbb{F}_{K^{\mathrm{ur}}})/p \to 0.$$

Since the residue field $\mathbb{F}_{K^{\mathrm{ur}}} = \overline{\mathbb{F}}_K$ is algebraically closed, $\overline{A}(\mathbb{F}_{K^{\mathrm{ur}}})/p = \overline{A}(\overline{\mathbb{F}}_K)/p = 0$ and hence the natural map $A_1(K^{\mathrm{ur}})/p \twoheadrightarrow A(K^{\mathrm{ur}})/p$ is surjective. This map gives the commutative diagram

$$\begin{tikzcd}
A(K^{\mathrm{ur}})/p \arrow[r, "\delta_A^{K^{\mathrm{ur}}}"] & ((K^{\mathrm{ur}})^\times/p)^{\oplus 2g} \\
A_1(K^{\mathrm{ur}})/p \arrow[u, two heads] \arrow[r, "\delta_{A_1}^{K^{\mathrm{ur}}}"] & ((K^{\mathrm{ur}})^\times/p)^{\oplus g} \arrow[u, "(\mathrm{id},1)"']
\end{tikzcd}$$

From this diagram, the image of $\delta_A^{K^{\mathrm{ur}}}$ is contained in $(\overline{U}_{K^{\mathrm{ur}}})^{\oplus g}$ as claimed. ∎

Next, we study the image of

$$\delta^K_{A_1} : A_1(K)/p \hookrightarrow H^1(K, A_1[p]) \simeq H^1(K, \mu_p)^{\oplus g} = (K^\times/p)^{\oplus g}.$$

LEMMA 3.3. $\mathrm{Im}(\delta^K_{A_1}) = (\overline{U}_K)^{\oplus g}$.

Proof. First, we show $\mathrm{Im}(\delta^K_{A_1}) \subset (\overline{U}_K)^{\oplus g}$. Consider the following commutative diagram:

$$\begin{tikzcd}
A_1(K)/p \arrow[r, "\delta^K_{A_1}"] \arrow[d] & (K^\times/p)^{\oplus g} \arrow[r, "v"] \arrow[d, "j"] & (\mathbb{Z}/p)^{\oplus g} \arrow[d, "\mathrm{id}"] \\
A_1(K^{\mathrm{ur}})/p \arrow[r, "\delta^{K^{\mathrm{ur}}}_{A_1}"] & ((K^{\mathrm{ur}})^\times/p)^{\oplus g} \arrow[r, "v"] & (\mathbb{Z}/p)^{\oplus g}
\end{tikzcd}$$

From Lemma 3.2(i), the composition $v \circ \delta^{K^{\mathrm{ur}}}_{A_1}$ is 0 in the above diagram. The composition $v \circ \delta^K_{A_1}$ is also 0 in the top sequence and hence $\mathrm{Im}(\delta_{A_1}) \subset (\overline{U}_K)^{\oplus g}$.

Next, we compare the orders of $\mathrm{Im}(\delta^K_{A_1})$ and $(\overline{U}_K)^{\oplus g}$. Since $\overline{U}_K = \mathrm{Ker}(v : K^\times/p \to \mathbb{Z}/p)$, the last claim indicates

$$\# A_1(K)/p \leq \#(\overline{U}_K)^{\oplus g} = p^{g([K:\mathbb{Q}_p]+1)}.$$

On the other hand, Mattuck's theorem [9] and the assumption $A[p] \subset A(K)$ say $A(K)/p \simeq (\mathbb{Z}/p)^{\oplus g([K:\mathbb{Q}_p]+2)}$. By the short exact sequence

$$0 \to \overline{A}(\mathbb{F}_K)[p] \to \overline{A}(\mathbb{F}_K) \xrightarrow{p} \overline{A}(\mathbb{F}_K) \to \overline{A}(\mathbb{F}_K)/p \to 0$$

of finite groups and $\overline{A}$ has ordinary reduction, we have

$$\#\overline{A}(\mathbb{F}_K)/p = \#\overline{A}(\mathbb{F}_K)[p] = p^g.$$

From the short exact sequence

$$A_1(K)/p \to A(K)/p \to \overline{A}(\mathbb{F}_K)/p \to 0,$$

we obtain the inequality $\# A_1(K)/p \geq p^{g([K:\mathbb{Q}_p]+1)}$. Therefore, the map $\delta^K_{A_1} : A_1(K)/p \xrightarrow{\simeq} (\overline{U}_K)^{\oplus g}$ is bijective. ∎

Proof of Proposition 3.1. First, we show

$$\mathrm{Im}(\delta^K_A) \subset (\overline{U}_K)^{\oplus g} \oplus \mathrm{Ker}(j : K^\times/p \to (K^{\mathrm{ur}})^\times/p)^{\oplus g}.$$

Consider the following commutative diagram:

$$\begin{tikzcd}
A(K)/p \arrow[r, "\delta^K_A"] \arrow[d] \arrow[drr, dotted, "\varphi"'] & (K^\times/p)^{\oplus 2g} \arrow[r, "v"] \arrow[d, "j"] & (\mathbb{Z}/p)^{\oplus 2g} \arrow[d, "\mathrm{id}"] \\
A(K^{\mathrm{ur}})/p \arrow[r, "\delta^{K^{\mathrm{ur}}}_A"] & ((K^{\mathrm{ur}})^\times/p)^{\oplus 2g} \arrow[r, "v"'] & (\mathbb{Z}/p)^{\oplus 2g}
\end{tikzcd} \tag{3.4}$$

From Lemma 3.2(ii), the image of the composition $j \circ \delta^K_A$ is contained in $(\overline{U}_{K^{\mathrm{ur}}})^{\oplus g}$. In particular, the image of φ the dotted arrow in the above diagram is 0 so that $\mathrm{Im}(\delta^K_A) \subset (\overline{U}_K)^{\oplus g} \oplus \mathrm{Ker}(j)^{\oplus g}$.

(i) From Mattuck's theorem [9] and the assumption $A[p]\subset A(k)$, $A(K)/p \simeq (\mathbb{Z}/p)^{\oplus g([K:\mathbb{Q}_p]+2)}$. On the other hand,

$$\begin{aligned}\mathrm{Ker}(K^\times/p \to (K^{\mathrm{ur}})^\times/p) &\simeq \mathrm{Ker}(H^1(K,\mu_p) \to H^1(K^{\mathrm{ur}},\mu_p))\\ &\simeq H^1(K^{\mathrm{ur}}/K,\mu_p) \simeq \mathbb{Z}/p,\end{aligned}$$

and hence

$$\#\big(\overline{U}_K \oplus \mathrm{Ker}(K^\times/p \to (K^{\mathrm{ur}})^\times/p)\big)^{\oplus g} = \#(K^\times/p)^{\oplus g} = p^{g([K:\mathbb{Q}_p]+2)}.$$

By counting the cardinalities, we obtain

$$\mathrm{Im}(\delta_A^K) = (\overline{U}_K)^{\oplus g} \oplus \mathrm{Ker}(K^\times/p \to (K^{\mathrm{ur}})^\times/p)^{\oplus g}.$$

(ii) Lemma 3.3 and the commutative diagram below give the assertion:

$$\begin{tikzcd} A_1(K)/p \arrow[r, "\delta_{A_1}^K"] \arrow[d] & (K^\times/p)^{\oplus g} \arrow[d, "(\mathrm{id},1)"] \\ A(K)/p \arrow[r, "\delta_A^K"] & (K^\times/p)^{\oplus 2g} \end{tikzcd}$$ ■

Now we define the sub Mackey functors $\mathscr{U},\mathscr{V} \subset \mathbb{G}_{\mathrm{m}}/p$ by

$$\begin{aligned}\mathscr{U}(K) &:= \overline{U}_K = \mathrm{Im}(U_K \to K^\times/p),\\ \mathscr{V}(K) &:= \mathrm{Ker}(j : K^\times/p \to (K^{\mathrm{ur}})^\times/p),\end{aligned} \tag{3.5}$$

for any finite extension K/k. Note that

$$\mathscr{V}(K) = \mathrm{Im}(U_K^{pe_0(K)} \to K^\times/p),$$

where $e_0(K) = e_K/(p-1)$ and e_K is the ramification index of $K/\mathbb{Q}_p$ [21, Rem. 3.2]. In fact, both of the subgroups of $K^\times/p$ are annihilators of $\overline{U}_K$ in the Hilbert symbol. By the fixed isomorphism (3.2), the following diagram is commutative:

$$\begin{tikzcd} A(L)/p \arrow[r, hook] \arrow[d, shift left, "N_{L/K}"] & H^1(L,A[p]) \arrow[r, "\simeq"] \arrow[d, shift left, "\mathrm{Cor}_{L/K}"] & (L^\times/p)^{\oplus g} \arrow[d, shift left, "N_{L/K}"] & \mathscr{U}(L)^{\oplus g} \oplus \mathscr{V}(L)^{\oplus g} \arrow[l, hook'] \arrow[d, shift left] \\ A(K)/p \arrow[r, hook] \arrow[u, shift left, "\mathrm{Res}_{L/K}"] & H^1(K,A[p]) \arrow[r, "\simeq"] \arrow[u, shift left, "\mathrm{Res}_{L/K}"] & (K^\times/p)^{\oplus g} \arrow[u, shift left] & \mathscr{U}(K)^{\oplus g} \oplus \mathscr{V}(K)^{\oplus g} \arrow[l, hook'] \arrow[u, shift left] \end{tikzcd}$$

for any finite extensions $L/K/k$. We obtain the following isomorphisms of Mackey functors:

Corollary 3.4. *There are isomorphisms*

$$A/p \simeq \mathscr{U}^{\oplus g} \oplus \mathscr{V}^{\oplus g} \quad \textit{and} \quad \widehat{A}/p \simeq \mathscr{U}^{\oplus g}$$

of Mackey functors.

4. Galois symbol map. Let A be an abelian variety over k which has good reduction. The formal group $\widehat{A}$ defines a Mackey functor by the associated group $\widehat{A}(K) := \widehat{A}(\mathfrak{m}_K)$ for a finite extension K/k. Note that $\widehat{A}(K) \simeq A_1(K)$ [4, Thm. C.2.6]. There is a short exact sequence of Mackey functors

$$0 \to \widehat{A} \to A \to \overline{\mathscr{A}} \to 0, \tag{4.1}$$

where $\overline{\mathscr{A}} = A/\widehat{A}$ is defined by exactness (in the abelian category of Mackey functors). The Mackey functor $\overline{\mathscr{A}}$ has the following description [13, Sect. 3, (3.3)]: For a finite extension K/k with residue field $\mathbb{F}_K$,

$$\overline{\mathscr{A}}(K) \simeq \overline{A}(\mathbb{F}_K). \tag{4.2}$$

For finite extensions $L/K/k$ with ramification index $e(L/K)$, the restriction $\mathrm{Res}_{L/K} : \overline{\mathscr{A}}(K) \to \overline{\mathscr{A}}(L)$ and the norm map $N_{L/K} : \overline{\mathscr{A}}(L) \to \overline{\mathscr{A}}(K)$ can be identified with the restriction $\overline{A}(\mathbb{F}_L) \to \overline{A}(\mathbb{F}_K)$ and $e(L/K)N_{\mathbb{F}_L/\mathbb{F}_K} : \overline{A}(\mathbb{F}_L) \to \overline{A}(\mathbb{F}_K)$ respectively.

For each $n \geq 1$, we denote

$$\widehat{\mathscr{M}_n} := (\mathbb{G}_{\mathrm{m}}/p^n \otimes \widehat{A}/p^n)(k) \quad \text{and} \quad \overline{\mathscr{M}}_n := (\mathbb{G}_{\mathrm{m}}/p^n \otimes \overline{\mathscr{A}}/p^n)(k).$$

By applying $\mathbb{G}_{\mathrm{m}}/p^n \otimes -$ (which is right exact) to (4.1), we have

$$\begin{tikzcd} & \widehat{\mathscr{M}_n} \arrow[r] \arrow[d, two heads] & (\mathbb{G}_{\mathrm{m}}/p^n \otimes A/p^n)(k) \arrow[r] \arrow[d, two heads] & \overline{\mathscr{M}}_n \arrow[r] \arrow[d, two heads, dashed] & 0 \\ 0 \arrow[r] & \widehat{\mathscr{K}_n} \arrow[r] & K(k;\mathbb{G}_{\mathrm{m}},A)/p^n \arrow[r] & \overline{\mathscr{K}}_n \arrow[r] & 0 \end{tikzcd} \tag{4.3}$$

where $\widehat{\mathscr{K}_n}$ is the image of the composition

$$\widehat{\mathscr{M}_n} \to (\mathbb{G}_{\mathrm{m}}/p^n \otimes A/p^n)(k) \to K(k;\mathbb{G}_{\mathrm{m}},A)/p^n$$

and $\overline{\mathscr{K}}_n$ is defined by the exactness of the lower sequence.

Special fiber

Lemma 4.1. *Let A be an abelian variety over k with good reduction. Then*

$$\overline{\mathscr{M}}_n = (\mathbb{G}_{\mathrm{m}}/p^n \otimes \overline{\mathscr{A}}/p^n)(k) \simeq \overline{A}(\mathbb{F})/p^n \quad \text{for any } n \in \mathbb{Z}_{\geq 1}.$$

Proof. We define a Mackey functor $\mathscr{Z}$ by $\mathscr{Z}(K) = \mathbb{Z}$ for each finite extension K/k and, for finite extensions $L/K/k$, the norm $N_{L/K} : \mathscr{Z}(L) \to \mathscr{Z}(K)$ is multiplication by the residue degree $f(L/K)$ of L/K and the restriction $\mathrm{Res}_{L/K} : \mathscr{Z}(K) \to \mathscr{Z}(L)$ is multiplication by the ramification index $e(L/K)$ of L/K. A morphism $v : \mathbb{G}_{\mathrm{m}} \to \mathscr{Z}$ of Mackey functors is defined by the valuation map $v : \mathbb{G}_{\mathrm{m}}(K) = K^{\times} \to \mathscr{Z}(K) = \mathbb{Z}$ for any finite extension K/k. As in (3.5), we define a Mackey functor $\mathscr{U}_n$ by $\mathscr{U}_n(K) := \mathrm{Im}(U_K \to K^{\times}/p^n)$, which fits into the short exact sequence

$$0 \to \mathscr{U}_n \to \mathbb{G}_{\mathrm{m}}/p^n \xrightarrow{v} \mathscr{Z}/p^n \to 0.$$

From the right exactness of $- \otimes \overline{\mathscr{A}}/p^n$ for $\overline{\mathscr{A}} = A/\widehat{A}$ (see (4.2)), we have

$$\mathscr{U}_n \otimes \overline{\mathscr{A}}/p^n \to \mathbb{G}_{\mathrm{m}}/p^n \otimes \overline{\mathscr{A}}/p^n \to \mathscr{Z}/p^n \otimes \overline{\mathscr{A}}/p^n \to 0. \tag{4.4}$$

First, we show $(\mathscr{Z}/p^n \otimes \overline{\mathscr{A}}/p^n)(k) \simeq \overline{A}(\mathbb{F})/p^n$. By identifying $\overline{\mathscr{A}}(K) \simeq \overline{A}(\mathbb{F}_K)$ (see (4.2)), define

$$\varphi : (\mathscr{Z}/p^n \otimes \overline{\mathscr{A}}/p^n)(k) \to \overline{A}(\mathbb{F})/p^n, \quad \{a, x\}_{K/k} \mapsto N_{\mathbb{F}_K/\mathbb{F}}(ax),$$

and

$$\psi : \overline{A}(\mathbb{F})/p^n \to (\mathscr{Z}/p^n \otimes \overline{\mathscr{A}}/p^n)(k), \quad x \mapsto \{1, x\}_{k/k}.$$

It is easy to see that these maps are well defined and $\varphi \circ \psi = \mathrm{id}$, where id is the identity map. To show $\psi \circ \varphi = \mathrm{id}$, take an element of the form $\{a, x\}_{K/k} \in (\mathscr{Z}/p^n \otimes \overline{\mathscr{A}}/p^n)(k)$. Let $k' \subset K$ be the maximal unramified subextension of K/k. We have

$$\begin{aligned}
&\psi \circ \varphi(\{a, x\}_{K/k}) = \{1, N_{\mathbb{F}_K/\mathbb{F}}(ax)\}_{k/k} \\
&= \{a, N_{k'/k}x\}_{k/k} \text{ (since } N_{k'/k} = N_{\mathbb{F}_K/\mathbb{F}} : \overline{\mathscr{A}}(K) = \overline{A}(\mathbb{F}_K) \to \overline{\mathscr{A}}(k) = \overline{A}(\mathbb{F})) \\
&= \{\mathrm{Res}_{k'/k}\, a, x\}_{k'/k} \quad (\text{by } (\mathbf{PF})) \\
&= \{a, x\}_{k'/k} \quad (\text{by } \mathrm{Res}_{k'/k}\, a = e(k'/k)a = a) \\
&= \{N_{K/k'}a, x\}_{k'/k} \quad (\text{by } N_{K/k'}(a) = f(K/k')a = a) \\
&= \{a, \mathrm{Res}_{K/k'}\, x\}_{K/k} \quad (\text{by } (\mathbf{PF})) \\
&= \{a, x\}_{K/k} \quad (\text{since } \mathrm{Res}_{K/k'} = \mathrm{id} : \overline{\mathscr{A}}(k') = \overline{A}(\mathbb{F}_K) \to \overline{\mathscr{A}}(K) = \overline{A}(\mathbb{F}_K)).
\end{aligned}$$

Consequently, $(\mathscr{Z}/p^n \otimes \overline{\mathscr{A}}/p^n)(k) \simeq \overline{A}(\mathbb{F})/p^n$.

Next, we prove $(\mathscr{U}_n \otimes \overline{\mathscr{A}}/p^n)(k) = 0$. Recall that $U_k^1 = 1 + \mathfrak{m}_k \subset U_k = O_k^\times$ induces $U_k/U_k^1 \simeq \mathbb{F}^\times$. The residue field $\mathbb{F}$ is finite, in particular, perfect, so $\mathscr{U}_n(k) = \mathrm{Im}(U_k^1 \to k^\times/p^n)$. By norm arguments, it is enough to show $\{a, x\}_{k/k} = 0$ in $(\mathscr{U}_n \otimes \overline{\mathscr{A}}/p^n)(k)$. For such an element $\{a, x\}_{k/k}$, there exists a finite unramified extension K/k such that $\mathrm{Res}_{K/k}(x) = p^n\xi$ for some $\xi \in \overline{\mathscr{A}}(K) \simeq \overline{A}(\mathbb{F}_K)$. Since the norm map $N_{K/k} : U_K^1 \to U_k^1$ is surjective [17, Chap. V, Prop. 3], one can find $\alpha \in \mathscr{U}_n(K)$ such that $N_{K/k}(\alpha) = a$. From this, we obtain

$$\begin{aligned}
\{a, x\}_{k/k} &= \{N_{K/k}(\alpha), x\}_{k/k} \\
&= \{\alpha, \mathrm{Res}_{K/k}(x)\}_{K/k} \quad (\text{by } (\mathbf{PF})) \\
&= \{\alpha, p^n\xi\}_{K/k} = 0.
\end{aligned}$$

This implies $(\mathscr{U}_n \otimes \overline{\mathscr{A}}/p^n)(k) = 0$. Finally, the short exact sequence (4.4) yields the assertion $\overline{\mathscr{M}}_n \simeq \overline{A}(\mathbb{F})/p^n$. ∎

Mackey product and the Somekawa K-group. We define

$$N := \max\{n \in \mathbb{Z}_{\geq 0} | A[p^n] \subset A(k)\}. \tag{4.5}$$

In the following, we assume that the abelian variety A satisfies the conditions (**Ord**) and (**Rat**) of the last section. From (**Rat**), we have $N \geq 1$. We fix isomorphisms $A[p^n] \simeq (\mu_{p^n})^{\oplus 2g}$ for all $n \leq N$ as follows: First, take an isomorphism $A[p^N] \simeq (\mu_{p^N})^{\oplus 2g}$ which makes the diagram

$$\begin{tikzcd} A_1[p^N] \arrow[r, hook] \arrow[d, "\simeq"] & A[p^N] \arrow[d, "\simeq"] \\ (\mu_{p^N})^{\oplus g} \arrow[r, hook, "(\mathrm{id},1)"] & (\mu_{p^N})^{\oplus g} \oplus (\mu_{p^N})^{\oplus g} \end{tikzcd}$$

commute as in (3.2). For each $1 \leq n < N$, we choose $A[p^n] \simeq (\mu_{p^n})^{\oplus 2g}$ so that the following diagram is commutative:

$$\begin{tikzcd} A[p^n] \arrow[r, hook] \arrow[d, "\simeq"] & A[p^{n+1}] \arrow[d, "\simeq"] \\ (\mu_{p^n})^{\oplus 2g} \arrow[r, hook] & (\mu_{p^{n+1}})^{\oplus 2g} \end{tikzcd} \tag{4.6}$$

To simplify the notation, put

$$\begin{aligned} \mathscr{M}_n &:= (\mathbb{G}_{\mathrm{m}} \otimes A)(k)/p^n \simeq (\mathbb{G}_{\mathrm{m}}/p^n \otimes A/p^n)(k), \\ \mathscr{H}_n &:= H^2(k, \mu_{p^n} \otimes A[p^n]). \end{aligned}$$

THEOREM 4.2. *For any $n \geq 1$, the Galois symbol map $s_{p^n}^M : \mathscr{M}_n \to \mathscr{H}_n$ (Def. 2.3) is bijective.*

Proof. The map $s_{p^n}^M$ is surjective [2, Thm. A.1]. We show that $s_{p^n}^M$ is injective by induction on n. First, we show that s_p^M is bijective. By the fixed isomorphism $A[p] \simeq (\mu_p)^{\oplus 2g}$ as in (4.6), we have the isomorphism

$$A/p \simeq \mathscr{U}^{\oplus g} \oplus \mathscr{V}^{\oplus g}$$

of Mackey functors (Cor. 3.4). The Mackey product $\mathscr{M}_1 = \mathbb{G}_{\mathrm{m}}/p \otimes A/p$ decomposes as

$$\mathscr{M}_1 = \mathbb{G}_{\mathrm{m}}/p \otimes (\mathscr{U}^{\oplus g} \oplus \mathscr{V}^{\oplus g}) \simeq (\mathbb{G}_{\mathrm{m}}/p \otimes \mathscr{U})^{\oplus g} \oplus (\mathbb{G}_{\mathrm{m}}/p \otimes \mathscr{V})^{\oplus g}.$$

Since the composition

$$\begin{aligned} (\mathbb{G}_{\mathrm{m}}/p \otimes \mathscr{V})(k) \to (\mathbb{G}_{\mathrm{m}}/p \otimes \mathscr{U})(k) \to (\mathbb{G}_{\mathrm{m}}/p \otimes \mathbb{G}_{\mathrm{m}}/p)(k) &\simeq K_2^M(k)/p \\ &\simeq H^2(k, \mu_p^{\otimes 2}) \end{aligned}$$

is bijective [5, Thm. 3.6], the Galois symbol map s_p^M is also bijective by the commutative diagram

$$\begin{tikzcd}
\mathscr{M}_1 \arrow[r, "s_p^M"] \arrow[d, "\simeq"] & \mathscr{H}_1 \arrow[d, "\simeq"] \\
(\mathbb{G}_{\mathrm{m}}/p \otimes \mathscr{U})(k)^{\oplus g} \oplus (\mathbb{G}_{\mathrm{m}}/p \otimes \mathscr{V})(k)^{\oplus g} \arrow[r, "\simeq"] & H^2(k, \mu_p^{\otimes 2})^{\oplus 2g}
\end{tikzcd}$$

To show that $s_{p^n}^M$ is injective, we consider the following commutative diagram with exact rows except possibly at $\mathscr{M}_n$ (cf. [13, proof of Lem. 4.2.2]):

$$\begin{tikzcd}
k^\times \otimes_{\mathbb{Z}} A[p] \arrow[r, "\psi"] \arrow[d, "\phi"] & \mathscr{M}_n \arrow[r, "p"] \arrow[d, hook, "s_{p^n}^M"] & \mathscr{M}_{n+1} \arrow[r] \arrow[d, "s_{p^{n+1}}^M"] & \mathscr{M}_1 \arrow[d, hook, "s_p^M"] \\
H^1(k, \mu_p \otimes A[p]) \arrow[r] & \mathscr{H}_n \arrow[r] & \mathscr{H}_{n+1} \arrow[r] & \mathscr{H}_1
\end{tikzcd}$$

Here, the vertical map ϕ is given by

$$k^\times \otimes_{\mathbb{Z}} A[p] \xrightarrow{\delta_{\mathbb{G}_{\mathrm{m}}} \otimes \mathrm{id}} H^1(k, \mu_p) \otimes_{\mathbb{Z}} H^0(k, A[p]) \xrightarrow{\cup} H^1(k, \mu_p \otimes A[p])$$

and ψ is induced from $A[p] \hookrightarrow A(k) \twoheadrightarrow A(k)/p^n$. By the fixed isomorphism $A[p] \simeq (\mu_p)^{\oplus 2g}$ of trivial Galois modules, the map ϕ becomes

$$k^\times \otimes_{\mathbb{Z}} A[p] \twoheadrightarrow (k^\times/p \otimes_{\mathbb{Z}} \mu_p)^{\oplus 2g} \simeq H^1(k, \mu_p^{\otimes 2})^{\oplus 2g} \simeq H^1(k, \mu_p \otimes A[p]).$$

In particular, ϕ is surjective. By diagram chase and the induction hypothesis, $s_{p^{n+1}}^M$ is injective. ∎

Corollary 4.3.

(i) *For any $n \geq 1$, we have $\mathscr{M}_n \simeq K(k; \mathbb{G}_{\mathrm{m}}, A)/p^n$.*
(ii) *For any $n \geq 1$, the Galois symbol map s_{p^n} is bijective.*
(iii) *For any $n \leq N$, we have $\mathscr{M}_n \simeq (\mathbb{Z}/p^n)^{\oplus 2g}$.*

Proof. As $s_{p^n}^M$ factors through s_{p^n}, assertions (i) and (ii) follow from Theorem 4.2. If $n \leq N$, we have

$$K(k; \mathbb{G}_{\mathrm{m}}, A)/p^n \xrightarrow{\simeq} \mathscr{H}_n \simeq H^2(k, \mu_{p^n}^{\otimes 2})^{\oplus 2g} \simeq (\mathbb{Z}/p^n)^{\oplus 2g}.$$

Assertion (iii) follows from (i). ∎

Note that $\mathscr{M}_n \simeq K(k; \mathbb{G}_{\mathrm{m}}, A)/p^n$, the middle vertical map in the diagram (4.3) is bijective (Cor. 4.3), and hence $\overline{\mathscr{M}}_n \simeq \overline{\mathscr{K}}_n$.

Corollary 4.4. *$\overline{\mathscr{K}}_n \simeq \overline{\mathscr{M}}_n \simeq \overline{A}(\mathbb{F})/p^n$.*

Proof. The isomorphism $\overline{\mathscr{M}}_n \simeq \overline{A}(\mathbb{F})/p^n$ follows from Lemma 4.1. ∎

Formal groups. Since the Mackey functor $\widehat{A}$ defined by the formal group of A satisfies Galois descent, we have the Galois symbol map (Def. 2.3) of the form

$$\widehat{s}_{p^n}^M : \widehat{\mathscr{M}_n} \simeq (\mathbb{G}_{\mathrm{m}} \otimes \widehat{A})(k)/p^n \to H^2(k, \mu_{p^n} \otimes \widehat{A}[p^n]) =: \widehat{\mathscr{H}_n}.$$

LEMMA 4.5. *For $n \le N$, we have the following:*

(i) *The Galois symbol map $\widehat{s}^M_{p^n} : \widehat{\mathscr{M}_n} \to \widehat{\mathscr{H}_n}$ is bijective.*

(ii) $\widehat{\mathscr{M}_n} \simeq \widehat{\mathscr{K}_n} \simeq (\mathbb{Z}/p^n)^{\oplus g}$.

Proof. (i) Consider the following commutative diagram with exact rows:

$$
\begin{tikzcd}
\widehat{\mathscr{M}_{n-1}} \arrow[r, "p"] \arrow[d, "\widehat{s}^M_{p^{n-1}}"] & \widehat{\mathscr{M}_n} \arrow[r] \arrow[d, "\widehat{s}^M_{p^n}"] & \widehat{\mathscr{M}_1} \arrow[r] \arrow[d, "\widehat{s}^M_p"] & 0 \\
\widehat{\mathscr{H}_{n-1}} \arrow[r] & \widehat{\mathscr{H}_n} \arrow[r] & \widehat{\mathscr{H}_1} \arrow[r] & 0
\end{tikzcd}
\tag{4.7}
$$

From the assumption $n \le N$, we have $\widehat{\mathscr{H}_n} \simeq H^2(k, \mu_{p^n}^{\otimes 2})^{\oplus g} \simeq (\mathbb{Z}/p^n)^{\oplus g}$. By counting the orders, the bottom left map in (4.7) is injective. By induction on n, it is enough to show that $\widehat{s}^M_p : \widehat{\mathscr{M}_1} \to \widehat{\mathscr{H}_1}$ is bijective. In this case, we have an isomorphism $\widehat{A}/p \xrightarrow{\simeq} \mathscr{U}^{\oplus g}$ of Mackey functors (Cor. 3.4). On the other hand, $\widehat{\mathscr{H}_1} \simeq H^2(k, \mu_p^{\otimes 2})^{\oplus g}$. The task is now reduced to showing that the composition

$$(\mathbb{G}_{\mathrm{m}}/p \otimes \mathscr{U})(k) \to (\mathbb{G}_{\mathrm{m}}/p \otimes \mathbb{G}_{\mathrm{m}}/p)(k) \xrightarrow{\simeq} H^2(k, \mu_p^{\otimes 2})$$

is bijective. This follows from [5, Theorem 3.6(i)].

(ii) The Galois symbol map $s_{p^n} : K(k; \mathbb{G}_{\mathrm{m}}, A)/p^n \to \mathscr{H}_n = H^2(k, \mu_{p^n} \otimes A[p^n])$ is bijective from Theorem 4.2 and $\mathscr{H}_n \simeq H^2(k, \mu_{p^n}^{\otimes 2})^{\oplus 2g} \simeq (\mathbb{Z}/p^n)^{\oplus 2g}$. From the commutative diagram (4.6), we have the following commutative diagram:

$$
\begin{tikzcd}
\widehat{\mathscr{H}_n} \arrow[r, "\iota"] \arrow[d, "\simeq"] & \mathscr{H}_n \arrow[d, "\simeq"] \\
H^2(k, \mu_{p^n}^{\otimes 2})^{\oplus g} \arrow[r, hook] & H^2(k, \mu_{p^n}^{\otimes 2})^{\oplus 2g}
\end{tikzcd}
$$

where $\widehat{\mathscr{H}_n} = H^2(k, \mu_{p^n} \otimes \widehat{A}[p^n])$. As the bottom map is an inclusion, the map ι is injective. Next, we consider the commutative diagram extended from (4.3):

$$
\begin{tikzcd}
\widehat{\mathscr{M}_n} \arrow[rr] \arrow[dr, "\simeq", "\widehat{s}^M_{p^n}"'] \arrow[dd, two heads] & & \mathscr{M}_n \arrow[dr, "s^M_{p^n}"] \arrow[dd, "\simeq"'] & \\
& \widehat{\mathscr{H}_n} \arrow[rr, hook, "\iota"] & & \mathscr{H}_n \\
\widehat{\mathscr{K}_n} \arrow[rr, hook] & & K(k; \mathbb{G}_{\mathrm{m}}, A)/p^n \arrow[ur, "s_{p^n}", "\simeq"'] &
\end{tikzcd}
$$

Since $\widehat{s}^M_{p^n} : \widehat{\mathscr{M}_n} \to \widehat{\mathscr{H}_n}$ is bijective, the left vertical map is bijective. Therefore, the induced map $\widehat{\mathscr{K}_n} \to \widehat{\mathscr{H}_n}$ is also bijective. ∎

To determine the structure of $\widehat{\mathscr{M}_n}$, we prepare the following key lemma.

Lemma 4.6 (cf. [3, proof of Thm. 3.14]). *Assume further the following condition:*

(**Ram**) *$k(\mu_{p^{N+1}})/k$ is a non-trivial and totally ramified extension.*

Fix a primitive p^Nth root of unity $\zeta \in \mu_{p^N}$. Then $\widehat{\mathscr{M}_1} = (\mathbb{G}_{\mathrm{m}}/p \otimes \widehat{A}/p)(k)$ is generated by elements of the form $\{\zeta, w\}_{k/k}$ for $w \in \widehat{A}(k)/p$.

Proof. Recall that the Hilbert symbol $(-,-) : k^\times \otimes k^\times \to \mu_p \simeq \mathbb{Z}/p$ satisfies

$$(x,y) = 0 \iff y \in N_{k(\sqrt[p]{x})/k}(k(\sqrt[p]{x})^\times) \quad \text{for } x, y \in k^\times \tag{4.8}$$

(see [22, Prop. 4.3]). By (**Ram**), there exists $y \in \overline{U}_k = \mathrm{Im}(U_k \to k^\times/p)$ such that $(\zeta, y) \neq 0$. In fact, putting $K = k(\sqrt[p]{\zeta}) = k(\mu_{p^{N+1}})$, we have $U_k/N_{K/k}U_K \simeq k^\times/N_{K/k}K^\times$ [17, Chap. V, Sect. 3] and local class field theory says $k^\times/N_{K/k}K^\times \simeq \mathrm{Gal}(K/k) \neq 0$ [17, Chap. XIII, Sect. 3]. Thus, there exists $y \in U_k \smallsetminus N_{K/k}U_K$ such that $(\zeta, y) \neq 0$ from (4.8). As $(\zeta, y) \neq 0$, y is non-trivial in $\overline{U}_k$. We use the same notation y as an element in $\overline{U}_k$. For each $1 \leq i \leq g$, put

$$y^{(i)} := (1, \dots, 1, \overset{\overset{i}{\vee}}{y}, 1, \dots, 1) \in (\overline{U}_k)^{\oplus g}$$

and we denote by $w^{(i)} \in \widehat{A}(k)/p$ the element corresponding to $y^{(i)}$ through the isomorphism $\widehat{A}(k)/p \simeq (\overline{U}_k)^{\oplus g}$ (Cor. 3.4). The Galois symbol map is compatible with the Hilbert symbol map [17, Chap. XIV, Sect. 2, Prop. 5] as the following commutative diagram indicates:

$$\begin{array}{ccccc} k^\times/p \otimes_{\mathbb{Z}} \widehat{A}(k)/p & \xrightarrow{\iota} & \widehat{\mathscr{M}_1} & \xrightarrow[\simeq]{\widehat{s}_p^M} & \widehat{\mathscr{K}_1} \\ \big\downarrow{\simeq} & & & & \big\downarrow{\simeq} \\ \left(k^\times/p \otimes_{\mathbb{Z}} \overline{U}_k\right)^{\oplus g} & & \xrightarrow{\quad(-,-)\quad} & & (\mathbb{Z}/p)^{\oplus g} \end{array} \tag{4.9}$$

Here, we identify $H^2(k, \mu_p^{\otimes 2}) \simeq \mathbb{Z}/p$, the map $\widehat{s}_p^M$ is bijective (Thm. 4.2), and the map ι is given by $\iota(x \otimes w) = \{x, w\}_{k/k}$. The image of $\zeta \otimes w^{(i)} \in k^\times/p \otimes_{\mathbb{Z}} \widehat{A}(k)/p$ in $(\mathbb{Z}/p)^{\oplus g}$ via the lower left corner in (4.9) is

$$\xi^{(i)} := (0, \dots, 0, \overset{\overset{i}{\vee}}{(\zeta, y)}, 0, \dots, 0) \in (\mathbb{Z}/p)^{\oplus g}.$$

The elements $\xi^{(i)}$ $(1 \leq i \leq g)$ generate $(\mathbb{Z}/p)^{\oplus g}$ and hence the symbols $\{\zeta, w^{(i)}\}_{k/k} = \iota(\zeta \otimes w^{(i)})$ $(1 \leq i \leq g)$ generate $\widehat{\mathscr{M}_1}$. ∎

Proposition 4.7. *Assume* (**Ram**). *Then*

$$\widehat{\mathscr{M}_n} \simeq \widehat{\mathscr{K}_n} \simeq (\mathbb{Z}/p^{\min\{N,n\}})^{\oplus g} \quad \textit{for any } n \geq 1.$$

Proof. From Lemma 4.5, we may assume $n \geq N$. There is a short exact sequence $\widehat{\mathscr{M}_1} \xrightarrow{p^n} \widehat{\mathscr{M}_{n+1}} \to \widehat{\mathscr{M}_n} \to 0$, where the first map is multiplication by p^n. By Lemma 4.5, we have $\widehat{\mathscr{K}_N} \simeq (\mathbb{Z}/p^N)^{\oplus g}$. From the above lemma and induction on $n \geq N$, we have $\widehat{\mathscr{M}_{n+1}} \simeq \widehat{\mathscr{M}_n} \simeq (\mathbb{Z}/p^N)^{\oplus g}$. The surjective homomorphisms

$$\widehat{\mathscr{M}_n} \twoheadrightarrow \widehat{\mathscr{K}_n} \twoheadrightarrow \widehat{\mathscr{K}_N} \simeq (\mathbb{Z}/p^N)^{\oplus g}$$

imply $\widehat{\mathscr{K}_n} \simeq (\mathbb{Z}/p^N)^{\oplus g}$. ∎

Proof of the main theorem. For an abelian variety A over k with good reduction (or split semi-stable reduction, more generally) we have

$$K(k; \mathbb{G}_{\mathrm{m}}, A) \simeq K(k; \mathbb{G}_{\mathrm{m}}, A)_{\mathrm{div}} \oplus K(k; \mathbb{G}_{\mathrm{m}}, A)_{\mathrm{fin}}$$

for a divisible group $K(k; \mathbb{G}_{\mathrm{m}}, A)_{\mathrm{div}}$ and a finite group $K(k; \mathbb{G}_{\mathrm{m}}, A)_{\mathrm{fin}}$ [13, Lem. 3.4 and proof of Thm. 4.5].

Theorem 4.8. *Let A be an abelian variety over k of dimension g. Assume that the conditions* (**Rat**), (**Ord**) *and* (**Ram**) *hold. Then*

$$K(k; \mathbb{G}_{\mathrm{m}}, A)_{\mathrm{fin}} \simeq (\mathbb{Z}/p^N)^{\oplus g} \oplus \overline{A}(\mathbb{F}).$$

Proof. For each m prime to p, $K(k; \mathbb{G}_{\mathrm{m}}, A)/m \simeq \overline{A}(\mathbb{F})/m$ (Prop. 2.6). First, we show

$$K(k; \mathbb{G}_{\mathrm{m}}, A)/p^n \simeq \widehat{\mathscr{K}_n} \oplus \overline{\mathscr{K}}_n$$

for each $n \geq 1$. The assertion is true for $n \leq N$ by Theorem 4.2 and Lemma 4.5. For $n > N$, consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathscr{K}_n} & \longrightarrow & K(k; \mathbb{G}_{\mathrm{m}}, A)/p^n & \longrightarrow & \overline{\mathscr{K}}_n & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \widehat{\mathscr{K}_{n+1}} & \longrightarrow & K(k; \mathbb{G}_{\mathrm{m}}, A)/p^{n+1} & \longrightarrow & \overline{\mathscr{K}}_{n+1} & \longrightarrow & 0
\end{array}$$

By the induction hypothesis, the top sequence splits. From Proposition 4.7, the left vertical map is bijective so that the lower sequence also splits.

From Corollary 4.4 and Proposition 4.7, we obtain

$$K(k; \mathbb{G}_{\mathrm{m}}, A)/p^n \simeq (\mathbb{Z}/p^{\min\{n,N\}})^{\oplus g} \oplus \overline{A}(\mathbb{F})/p^n$$

for any n. By taking the limit $\varprojlim$, we have

$$K(k; \mathbb{G}_{\mathrm{m}}, A)_{\mathrm{fin}} \simeq (\mathbb{Z}/p^N)^{\oplus g} \oplus \overline{A}(\mathbb{F}). \ ∎$$

Applying the above theorem to the Jacobian variety $\mathrm{Jac}(X)$, we obtain the following corollary as noted in Introduction.

Corollary 4.9. *Let X be a projective smooth curve over k with $X(k) \neq \emptyset$. Assume the conditions* (**Rat**), (**Ord**) (*at the beginning of Sect.* 3), *and* (**Ram**) (*in Lem.* 4.6) *for the Jacobian variety $J = \mathrm{Jac}(X)$ associated to X. Then*

$$V(X)_{\mathrm{fin}} \simeq (\mathbb{Z}/p^N)^{\oplus g} \oplus \overline{J}(\mathbb{F}),$$

where $g = \dim J$ and $N = \max\{n | J[p^n] \subset J(k)\}$.

To end this section, we refer to the case where the base field has low ramification.

THEOREM 4.10 (Kato–Saito, Yoshida). *Let X be a projective smooth curve over k with $X(k) \neq \emptyset$. Assume that X has good reduction and $e_k < p-1$, where e_k is the absolute ramification index of k. Then*

$$V(X)_{\mathrm{fin}} \simeq \overline{J}(\mathbb{F}).$$

Proof. Let $\mathscr{X}$ be the regular model over O_k of X and $\overline{\mathscr{X}} := \mathscr{X} \otimes_{O_k} \mathbb{F}$ the special fiber of $\mathscr{X}$. The specialization map gives a canonical surjection $\mathrm{sp} : \pi_1^{\mathrm{ab}}(X) \to \pi_1^{\mathrm{ab}}(\overline{\mathscr{X}})$. Now, we consider the short exact sequence

$$0 \to \pi_1^{\mathrm{ab}}(X)^{\mathrm{geo}}_{\mathrm{ram}} \to \pi_1^{\mathrm{ab}}(X)^{\mathrm{geo}} \xrightarrow{\mathrm{sp}} \pi_1^{\mathrm{ab}}(\overline{\mathscr{X}})^{\mathrm{geo}} \to 0,$$

where $\pi_1^{\mathrm{ab}}(\overline{\mathscr{X}})^{\mathrm{geo}} := \mathrm{Ker}(\pi_1^{\mathrm{ab}}(\overline{X}) \to \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}))$, and $\pi_1^{\mathrm{ab}}(X)^{\mathrm{geo}}_{\mathrm{ram}}$ is defined by exactness. Here, $X(k) \neq \emptyset$ implies that the specialization $\mathrm{sp} : \pi_1^{\mathrm{ab}}(X)^{\mathrm{geo}} \to \pi_1^{\mathrm{ab}}(\overline{\mathscr{X}})^{\mathrm{geo}}$ on the geometric parts is surjective. Under the assumptions, it is known that $\pi_1^{\mathrm{ab}}(X)^{\mathrm{geo}}_{\mathrm{ram}} = 0$ ([7, Prop. 7], [25, Thms. 3.2, 4.1]). In particular, there are isomorphisms

$$V(X)_{\mathrm{fin}} \simeq \pi_1^{\mathrm{ab}}(X)^{\mathrm{geo}} \simeq \pi_1^{\mathrm{ab}}(\overline{\mathscr{X}})^{\mathrm{geo}} \simeq \overline{J}(\mathbb{F}),$$

where the first isomorphism follows from the reciprocity map τ (see (2.8)) and the last one comes from the class field theory of curves over finite fields. ∎

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Abstract (will appear on the journal's web site only)

We study the Galois symbol map associated to the multiplicative group and an abelian variety which has good ordinary reduction over a p-adic field. As a byproduct, one can calculate the "class group" in the sense of the class field theory for curves over a p-adic field.